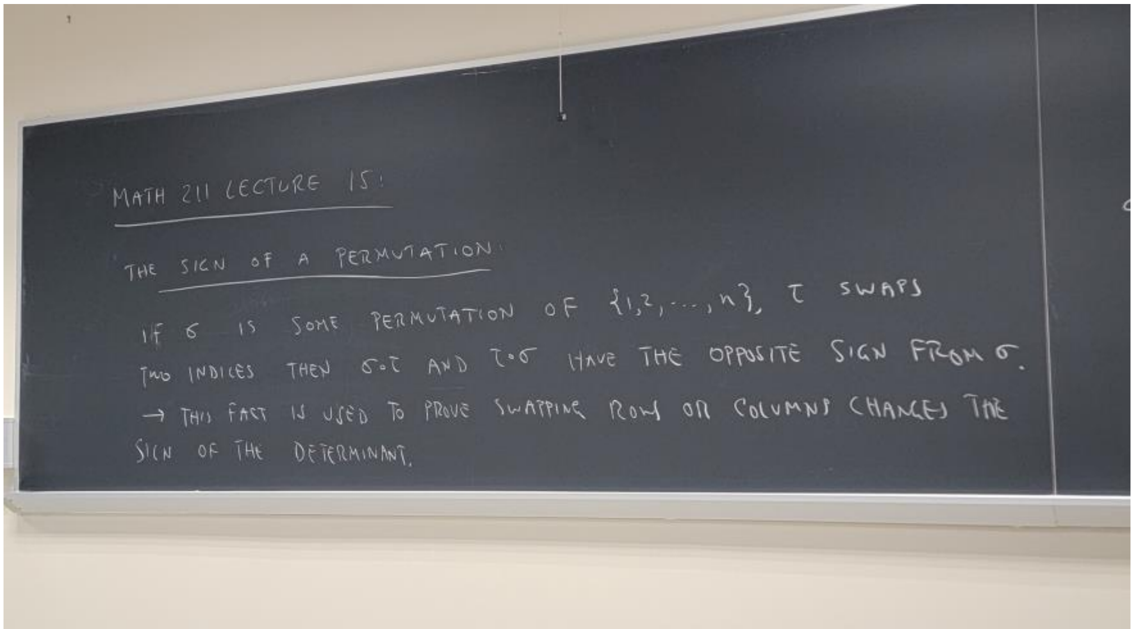
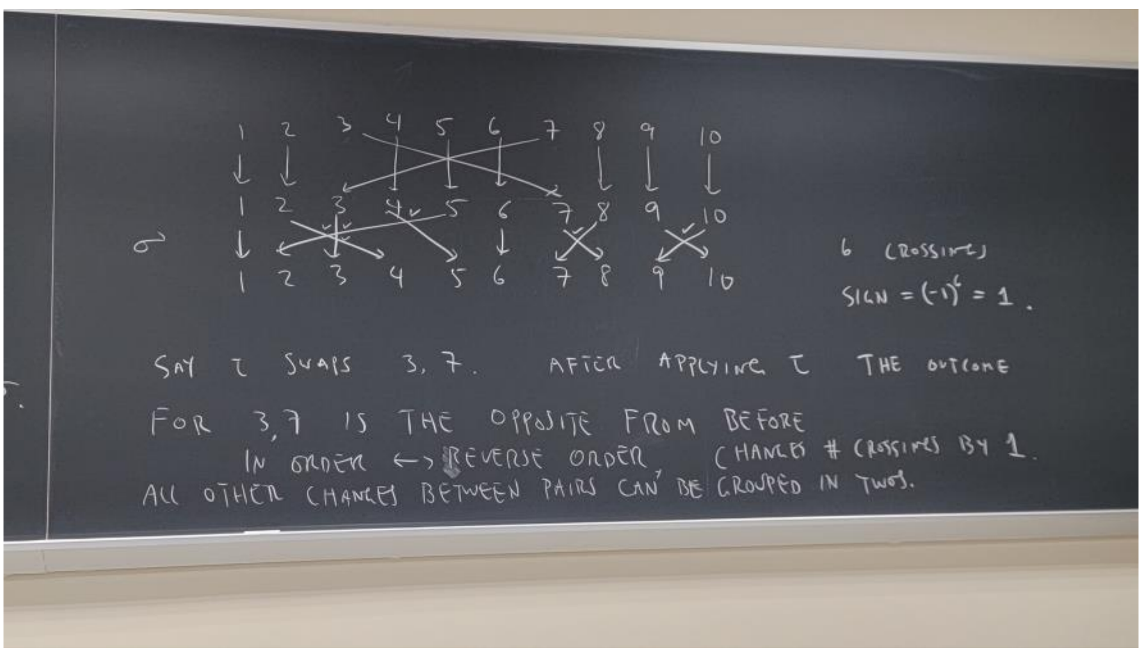
3/23/23

Monday, March 27, 2023 7:05 PM





THEOREM. LET O BE AN ORTHOGONAL MATRIX,
THAT IS $O^T \cdot O = O \cdot O^T = 1$. THEN $\det(O) = \pm 1$.

PROOF. $O^T O = I_n \Rightarrow \det(O^T) \cdot \det(O) = \det(O)^2 = 1$

$$\Rightarrow \det(O) = \pm 1$$

DEFINITION: WE SAY AN ORTHOGONAL MATRIX $n \times n$ OF DETERMINANT 1 IS A ROTATION.
THESE MATRICES ARE ORIENTATION PRESERVING, DETERMINANT -1, ORIENTATION REVERSING.

EXAMPLE: FOR 2×2 MATRICES, THE ORTHOGONAL MATRICES ARE ROTATIONS AND REFLECTIONS.

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

ROTATION BY θ.

REFLECTION: $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

IN LINE ANGLE $\theta/2$

$\det = -1$

$$\det(A) = \underbrace{\det(Q)}_{\pm 1,\ \text{ORTHOGONAL.}} \cdot \det(R)$$

$$\det(R) = \|\underline{v}_1\| \cdot \|\underline{v}_2^{\perp}\| \cdot \ \ldots \ \|\underline{v}_n^{\perp}\|.$$

$$|\det A| = \|\underline{v}_1\| \cdot \|\underline{v}_2^{\perp}\| \ldots \ \|\underline{v}_n^{\perp}\|.$$

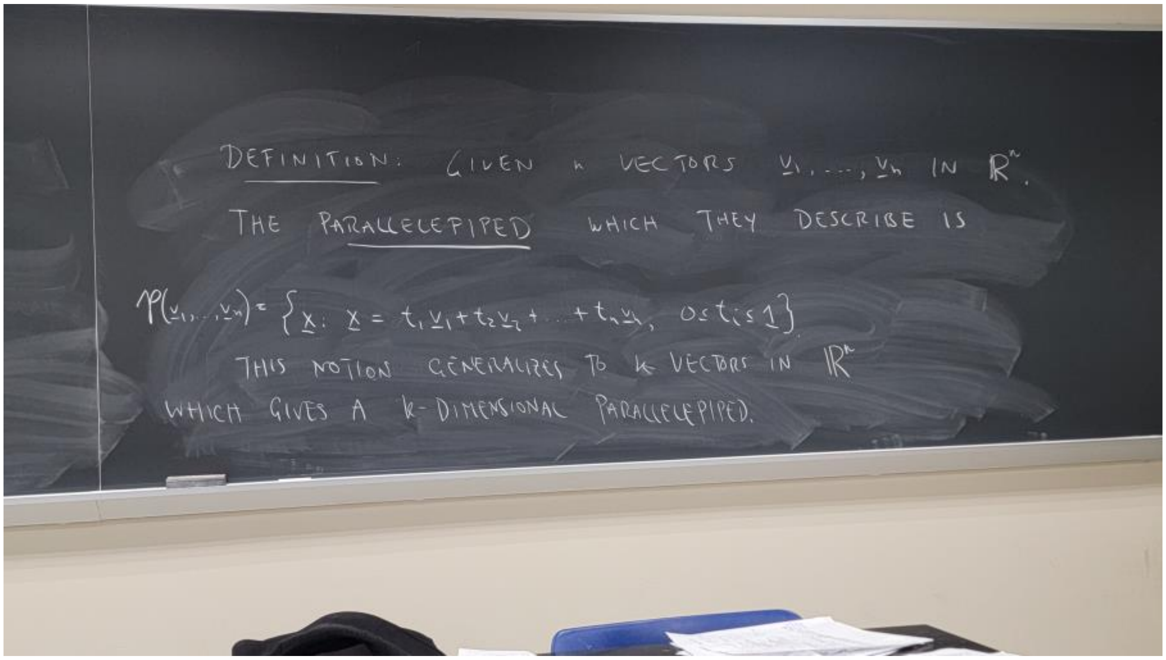
REMARK: THE DETERMINANT DESCRIBES THE CHANGE TO AREA OR VOLUME UNDER LINEAR TRANSFORMATION

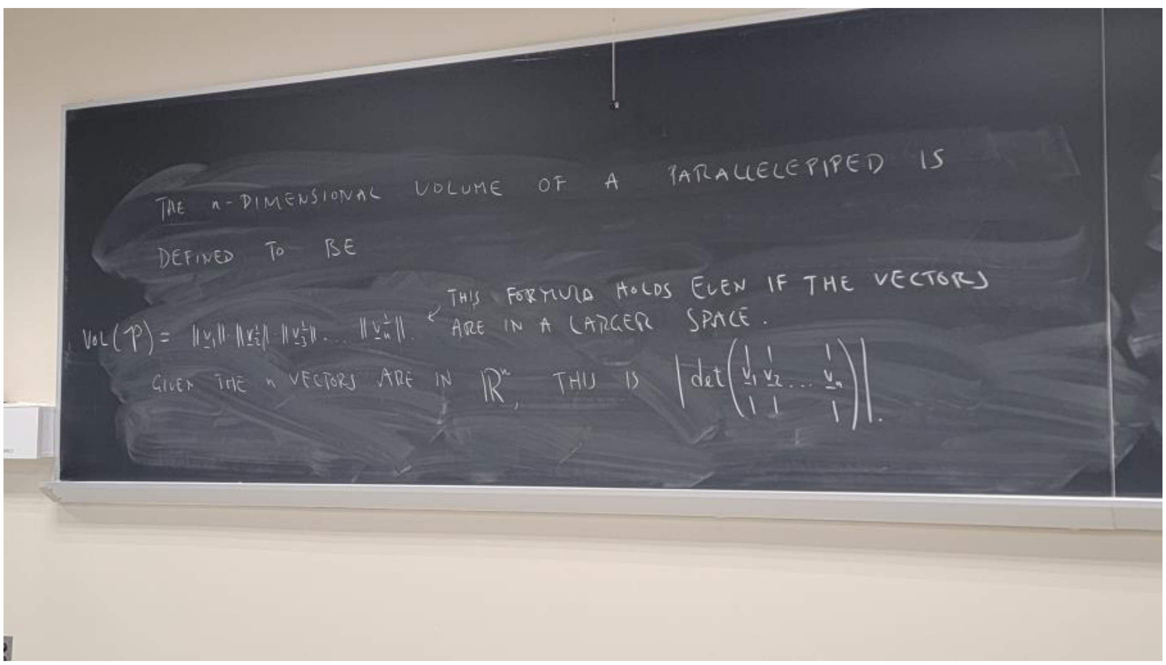
$$\underline{x} \longmapsto A\underline{x}.$$

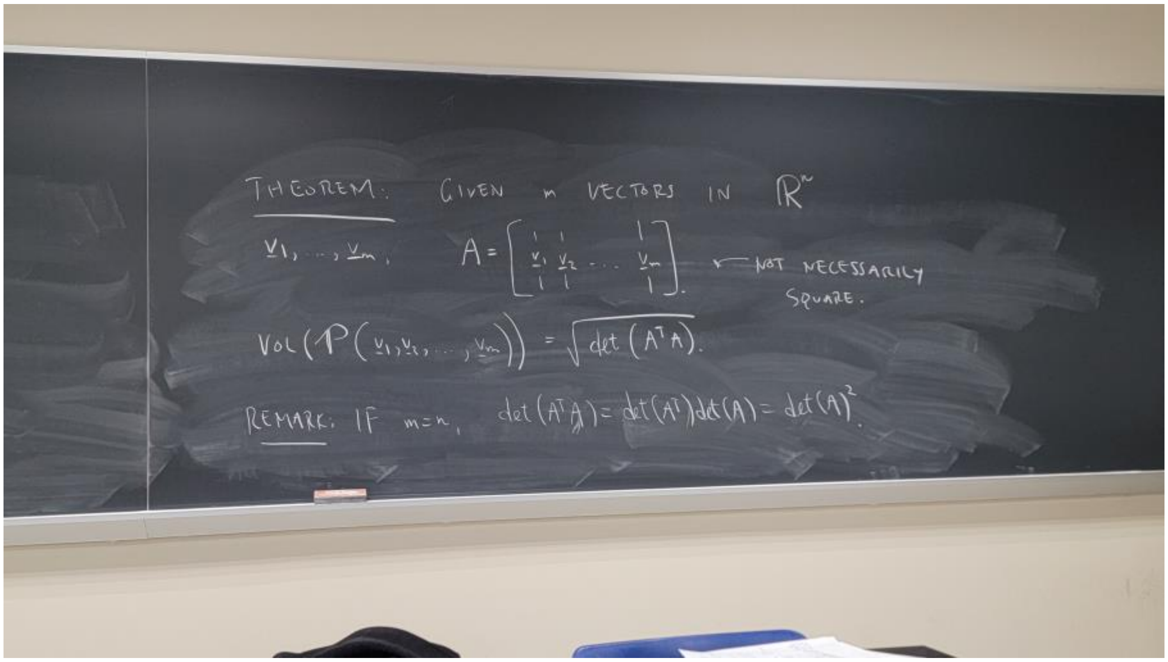
PARALLELOGRAM.

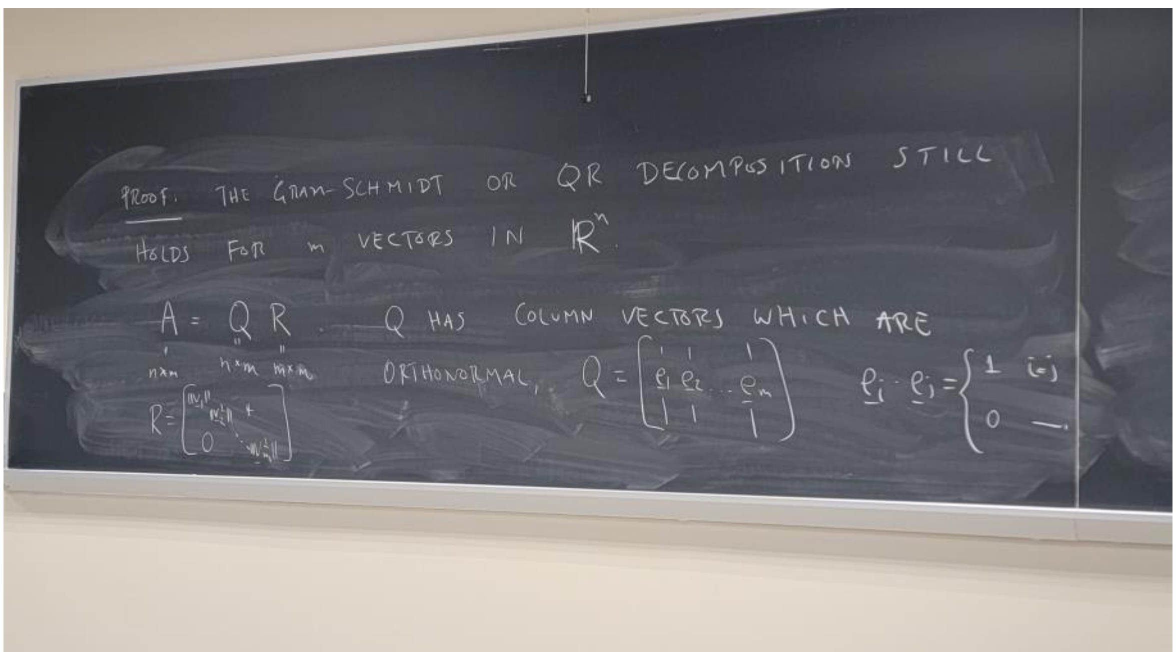
AREA. BASE × HEIGHT

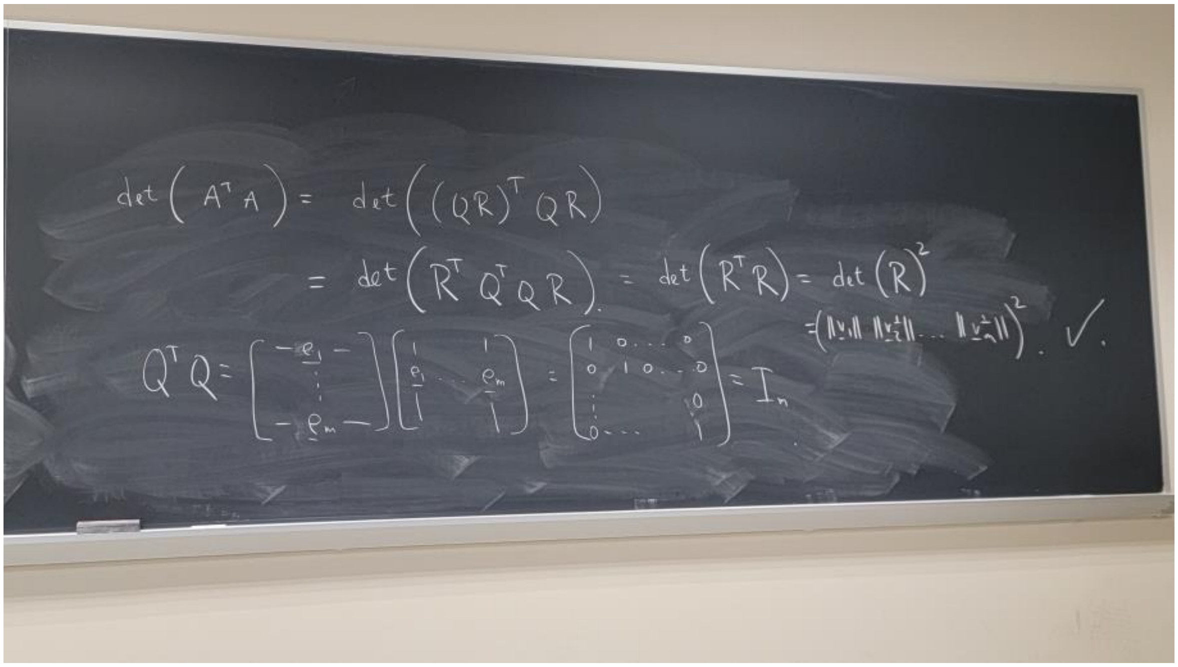
$$= \|\underline{v}_1\| \cdot \|\underline{v}_2^{\perp}\| = \left|\det\begin{pmatrix} | & | \\ \underline{v}_1 & \underline{v}_2 \\ | & | \end{pmatrix}\right|.$$

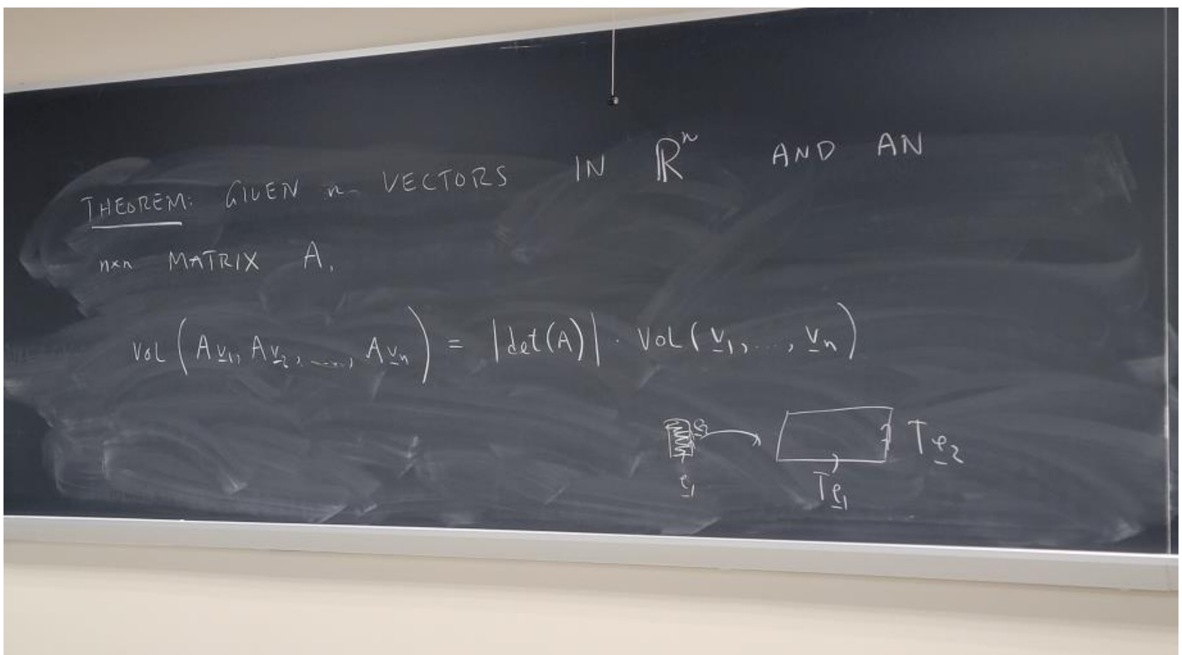

DEFINITION: GIVEN n VECTORS $\underline{v}_1, \dots, \underline{v}_n$ IN $\mathbb{R}^n$.
THE PARALLELEPIPED WHICH THEY DESCRIBE IS
$\mathcal{P}(\underline{v}_1, \dots, \underline{v}_n) = \{ \underline{x} : \underline{x} = t_1 \underline{v}_1 + t_2 \underline{v}_2 + \dots + t_n \underline{v}_n,\ 0 \le t_i \le 1 \}$
THIS NOTION GENERALIZES TO k VECTORS IN $\mathbb{R}^n$
WHICH GIVES A k-DIMENSIONAL PARALLELEPIPED.

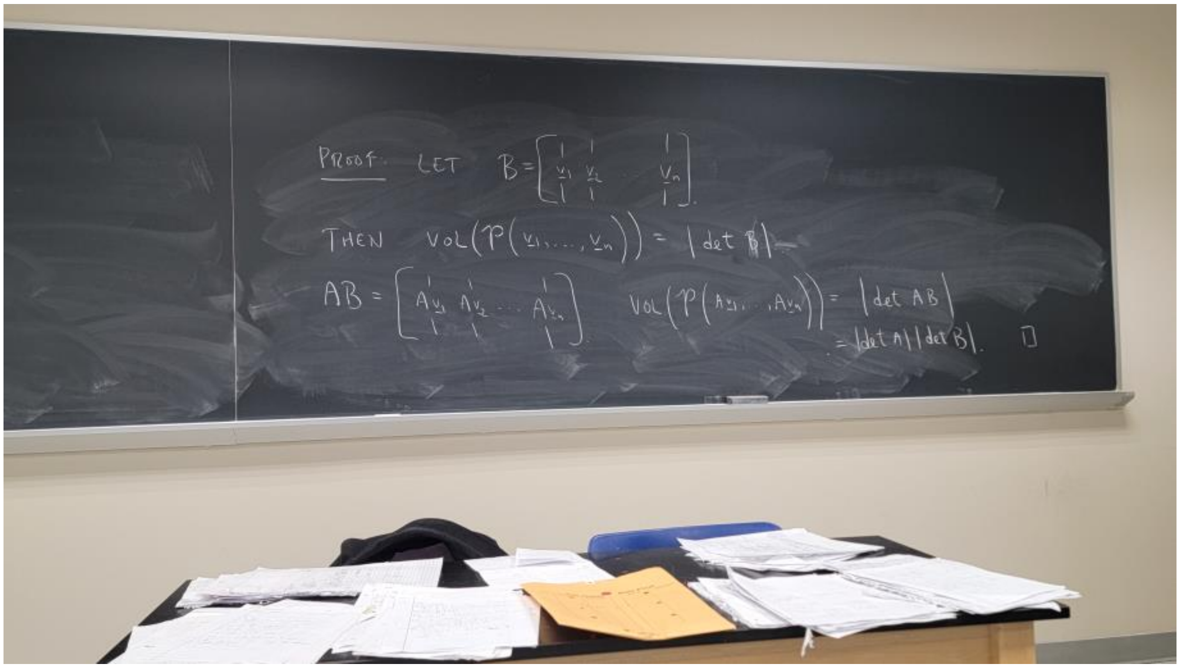

THE n-DIMENSIONAL VOLUME OF A PARALLELEPIPED IS
DEFINED TO BE
$\mathrm{Vol}(\mathcal{P}) = \|\underline{v}_1\| \|\underline{v}_2^{\perp}\| \|\underline{v}_3^{\perp}\| \dots \|\underline{v}_n^{\perp}\|.$
THIS FORMULA HOLDS EVEN IF THE VECTORS ARE IN A LARGER SPACE.
GIVEN THE n VECTORS ARE IN $\mathbb{R}^n$, THIS IS $\left| \det \begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \dots & \underline{v}_n \\ | & | & & | \end{pmatrix} \right|$.

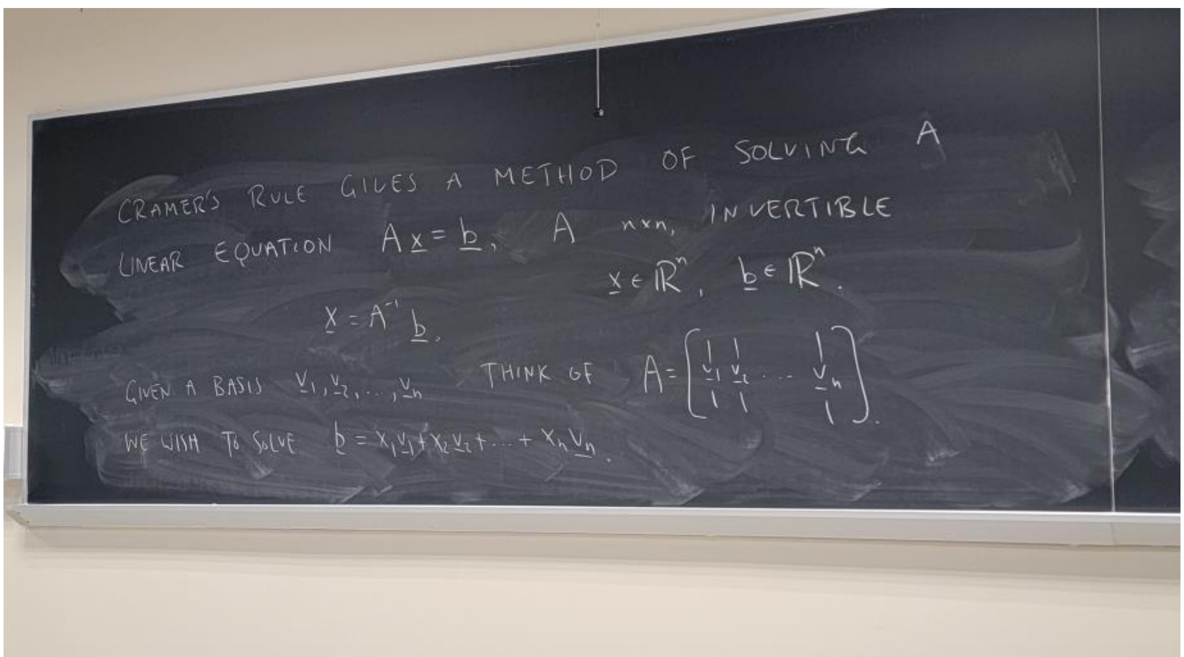

THEOREM: GIVEN m VECTORS IN $\mathbb{R}^n$
$\underline{v}_1, \ldots, \underline{v}_m$. $A = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_m \\ | & | & & | \end{bmatrix}$.
NOT NECESSARILY SQUARE.
$\mathrm{Vol}(\mathcal{P}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m)) = \sqrt{\det(A^T A)}$.
REMARK: IF $m=n$, $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2$.


PROOF. THE GRAM-SCHMIDT OR QR DECOMPOSITION STILL
HOLDS FOR m VECTORS IN $\mathbb{R}^n$.
$\underset{n \times m}{A} = \underset{n \times m}{Q}\ \underset{m \times m}{R}$. Q HAS COLUMN VECTORS WHICH ARE
ORTHONORMAL, $Q = \begin{bmatrix} | & | & & | \\ \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_m \\ | & | & & | \end{bmatrix}$
$\underline{e}_i \cdot \underline{e}_j = \begin{cases} 1 & i=j \\ 0 & \end{cases}$
$R =$


$\det(A^T A) = \det((QR)^T QR)$
$= \det(R^T Q^T Q R) = \det(R^T R) = \det(R)^2$
$= (\|v_1\| \, \|v_2^\perp\| \cdots \|v_n^\perp\|)^2$. ✓
$Q^T Q = \begin{bmatrix} - e_1 - \\ \vdots \\ - e_m - \end{bmatrix} \begin{bmatrix} | & & | \\ e_1 & \cdots & e_m \\ | & & | \end{bmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & & 0 \\ 0 & \cdots & & 1 \end{pmatrix} = I_n$


THEOREM: GIVEN n VECTORS IN $\mathbb{R}^n$ AND AN
$n \times n$ MATRIX A,
$\mathrm{vol}(A\underline{v}_1, A\underline{v}_2, \ldots, A\underline{v}_n) = |\det(A)| \cdot \mathrm{vol}(\underline{v}_1, \ldots, \underline{v}_n)$
$\underline{e}_2$
$\underline{e}_1$
$T\underline{e}_2$
$T\underline{e}_1$


PROOF. LET $B = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix}$.
THEN $\mathrm{VOL}(\mathcal{P}(\underline{v}_1, \ldots, \underline{v}_n)) = |\det B|$
$AB = \begin{bmatrix} | & | & & | \\ A\underline{v}_1 & A\underline{v}_2 & \cdots & A\underline{v}_n \\ | & | & & | \end{bmatrix}$ $\mathrm{VOL}(\mathcal{P}(A\underline{v}_1, \ldots, A\underline{v}_n)) = |\det AB|$
$= |\det A| |\det B|$. □


CRAMER'S RULE GIVES A METHOD OF SOLVING A
LINEAR EQUATION $A\underline{x} = \underline{b}$, A $n \times n$, INVERTIBLE
$\underline{x} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^n$.
$\underline{x} = A^{-1}\underline{b}$.
GIVEN A BASIS $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ THINK OF $A = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix}$.
WE WISH TO SOLVE $\underline{b} = x_1\underline{v}_1 + x_2\underline{v}_2 + \ldots + x_n\underline{v}_n$.

TO CALCULATE x_i, REPLACE $\underline{v}_i$ WITH $\underline{b}$ AND TAKE THE DETERMINANT.

$$\det\begin{bmatrix} | & | & & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{b} & \cdots & \underline{v}_n \\ | & | & & | & & | \end{bmatrix} = \det\begin{bmatrix} | & | & & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & (x_1\underline{v}_1+\cdots+x_n\underline{v}_n) & \cdots & \underline{v}_n \\ | & | & & | & & | \end{bmatrix}$$

iTH COLUMN. iTH COLUMN

BY LINEARITY IN THE iTH COLUMN, THE det IS THE SUM OF n DETERMINANTS, ALL BUT ONE OF WHICH IS 0.

$$= x_i \det\begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix} = x_i \det A, \qquad x_i = \frac{\det\begin{bmatrix} | & & | & | & & | \\ \underline{v}_1 & \cdots & \underline{v}_{i-1} & \underline{b} & \cdots & \underline{v}_n \\ | & & | & | & & | \end{bmatrix}}{\det A}.$$

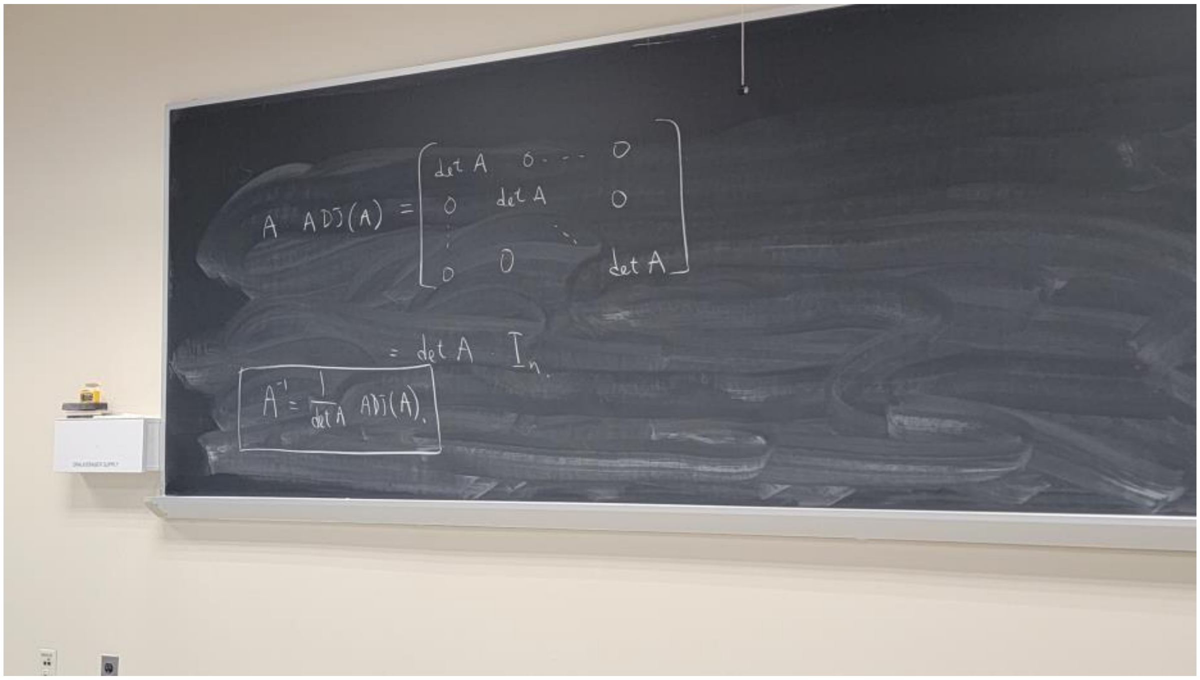
CRAMER'S RULE GIVES A METHOD OF CALCULATING THE INVERSE:

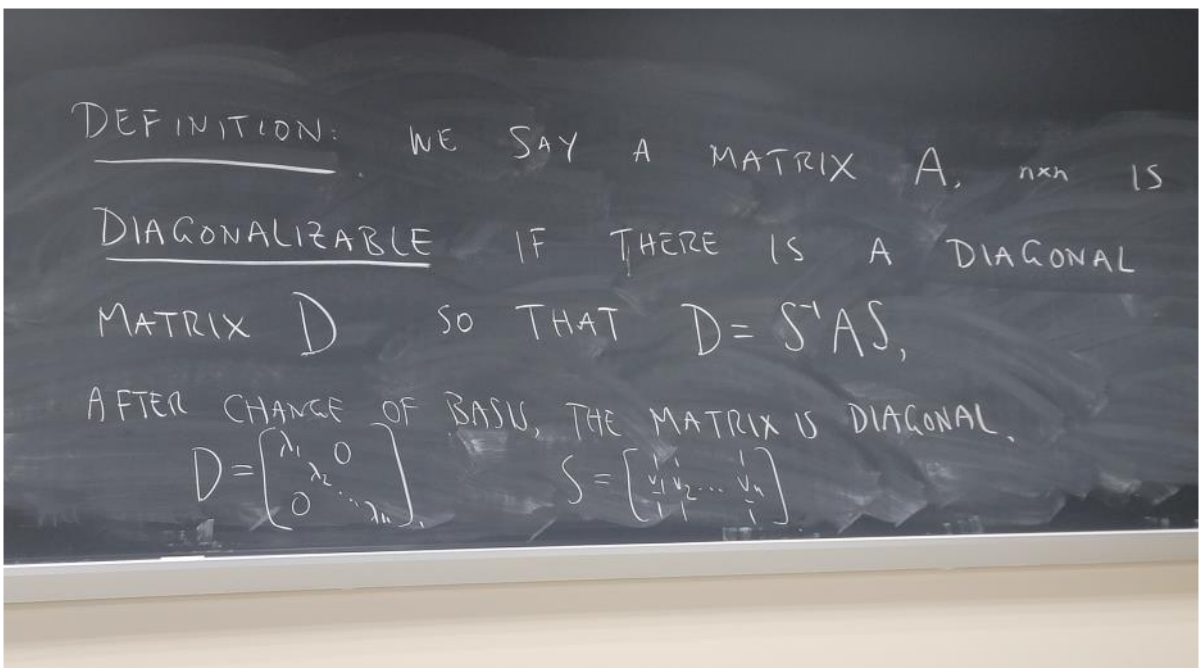
$$A = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix}. \qquad A^{-1} = \begin{bmatrix} | & | & & | \\ \underline{w}_1 & \underline{w}_2 & \cdots & \underline{w}_n \\ | & | & & | \end{bmatrix}. \qquad A \cdot A^{-1} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$$

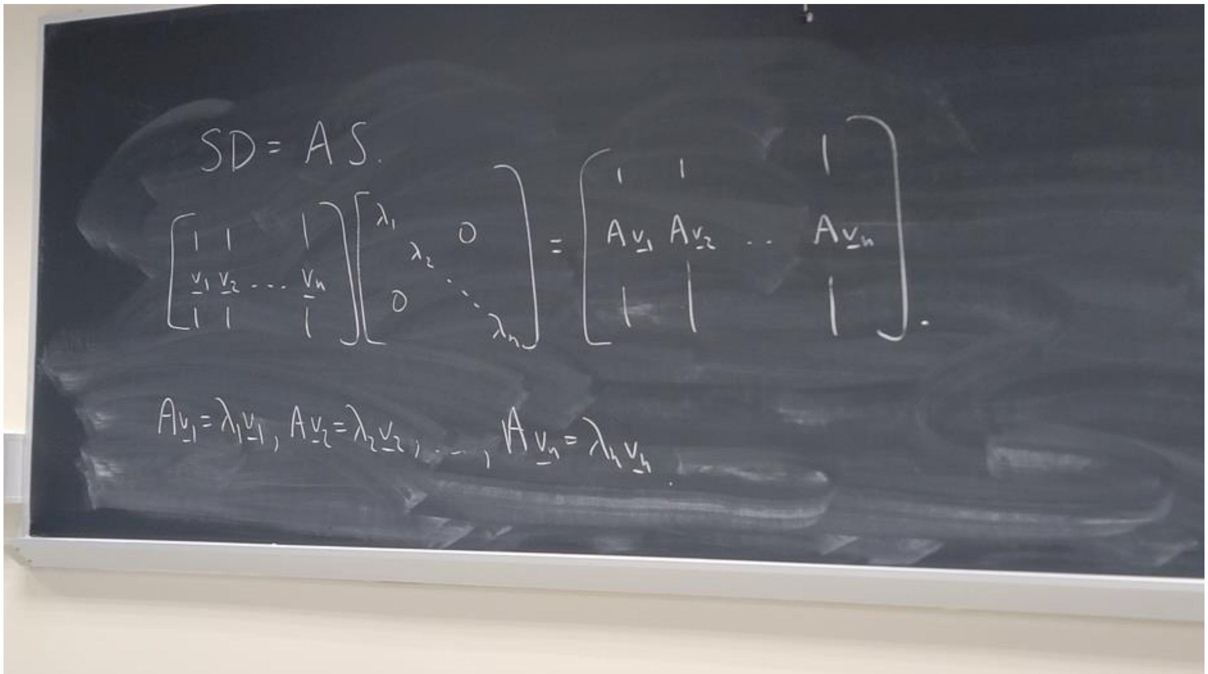
SOLVE
$$A\underline{w}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\underline{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad A\underline{w}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

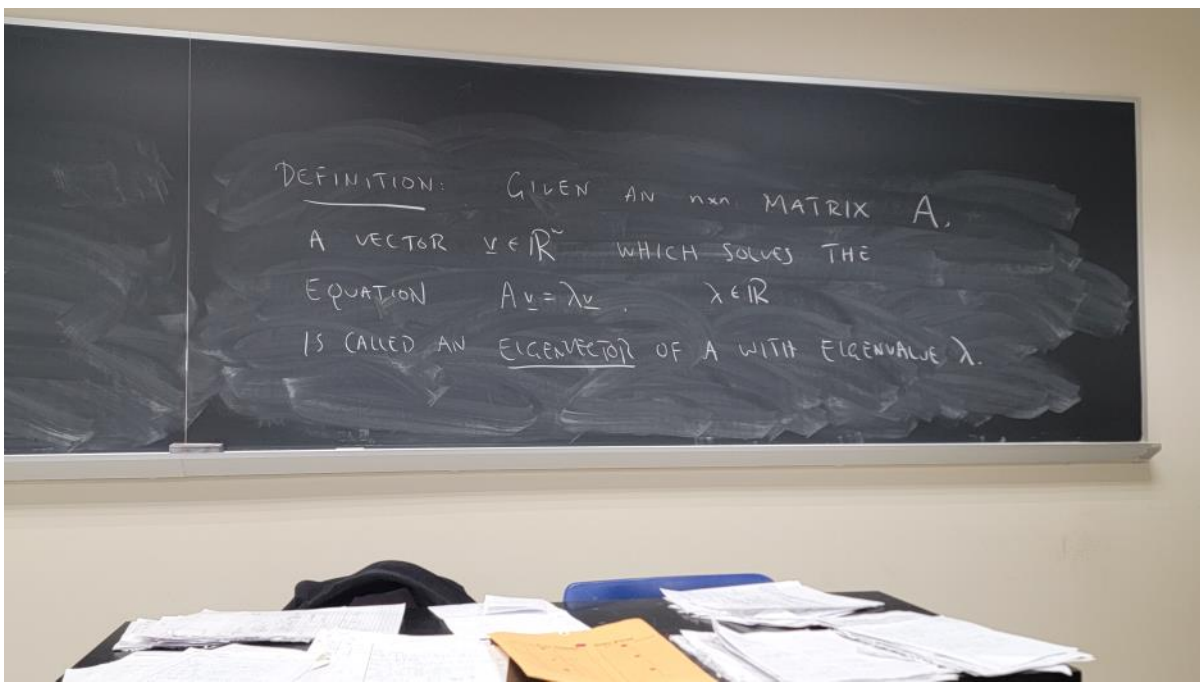
$$\det \begin{array}{c} \begin{bmatrix} | & | & & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{e}_j & \cdots & \underline{v}_n \\ | & | & & | & & | \end{bmatrix} \\ \text{column } i \end{array} = (-1)^{i+j} \det A_{ji}.$$

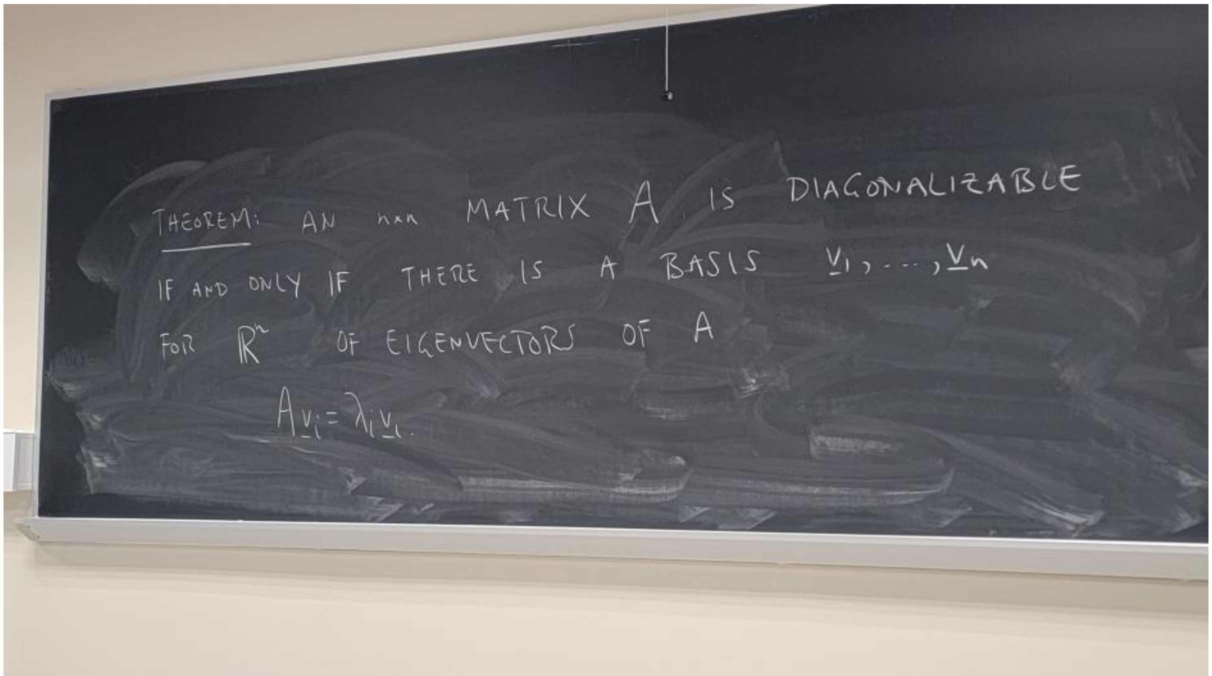
THE MATRIX B WITH ij ENTRY $b_{ij} = (-1)^{i+j} \det A_{ji}$.
IS CALLED THE <u>ADJUGATE</u> MATRIX OF A.

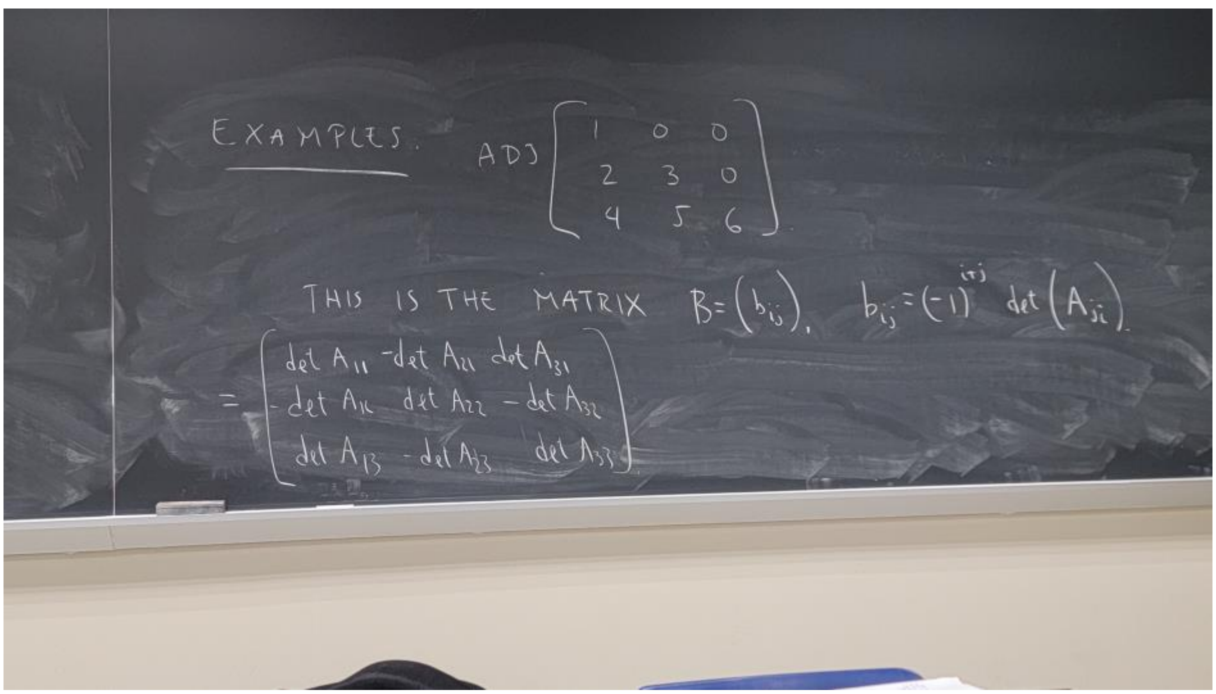

$A \cdot ADJ(A) = \begin{bmatrix} \det A & 0 & \cdots & 0 \\ 0 & \det A & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \det A \end{bmatrix}$
$= \det A \cdot I_n.$
$\boxed{A^{-1} = \frac{1}{\det A} ADJ(A).}$

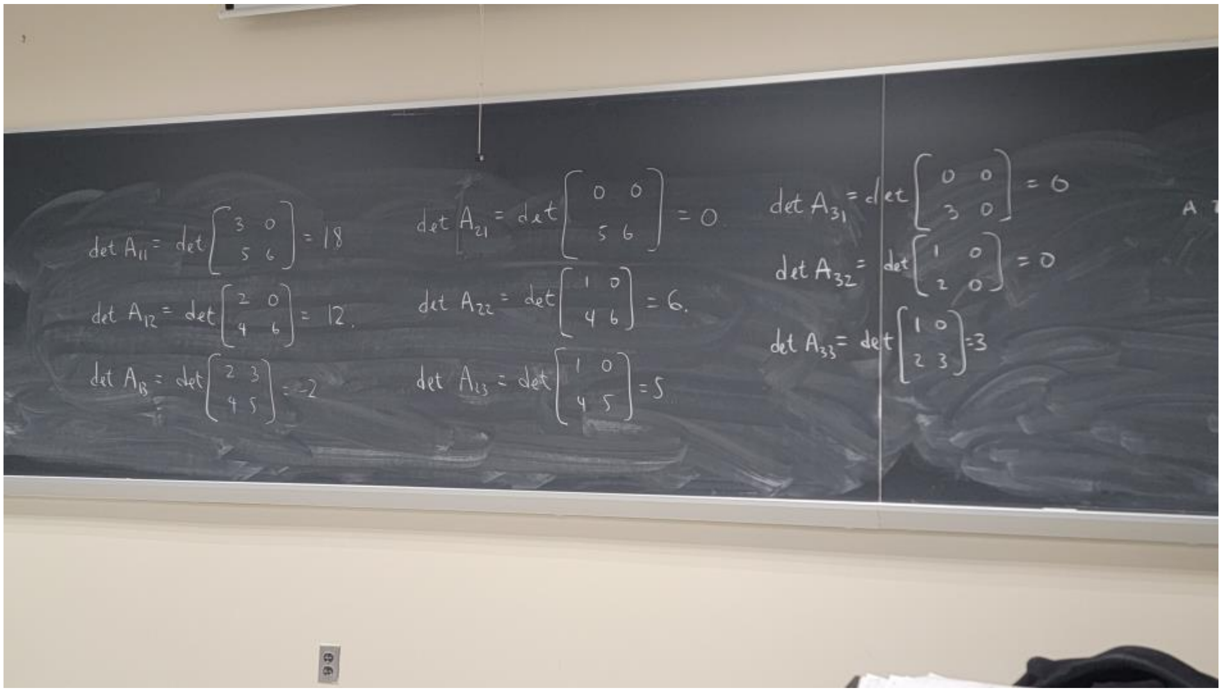

DEFINITION: WE SAY A MATRIX A, $n \times n$ IS DIAGONALIZABLE IF THERE IS A DIAGONAL MATRIX D SO THAT $D = S^{-1}AS$,
AFTER CHANGE OF BASIS, THE MATRIX IS DIAGONAL.
$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$, $S = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix}$.

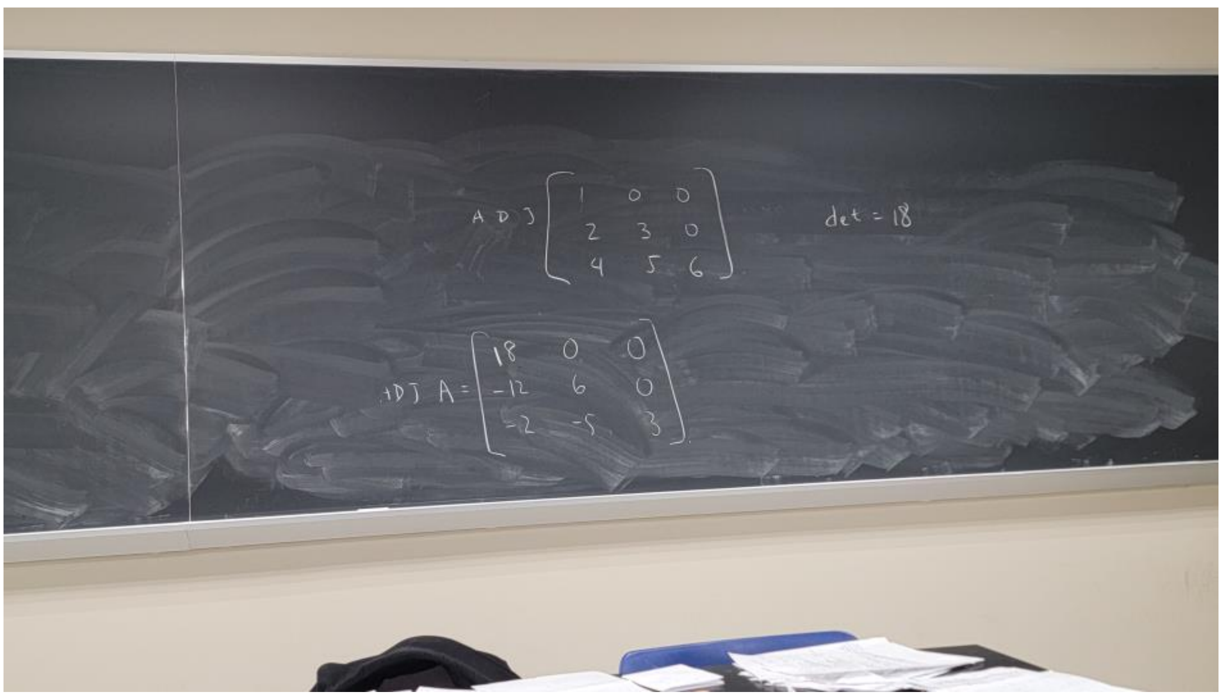

$SD = AS.$
$\begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\underline{v}_1 & A\underline{v}_2 & \cdots & A\underline{v}_n \\ | & | & & | \end{bmatrix}.$
$A\underline{v}_1 = \lambda_1 \underline{v}_1,\ A\underline{v}_2 = \lambda_2 \underline{v}_2, \ldots, A\underline{v}_n = \lambda_n \underline{v}_n.$

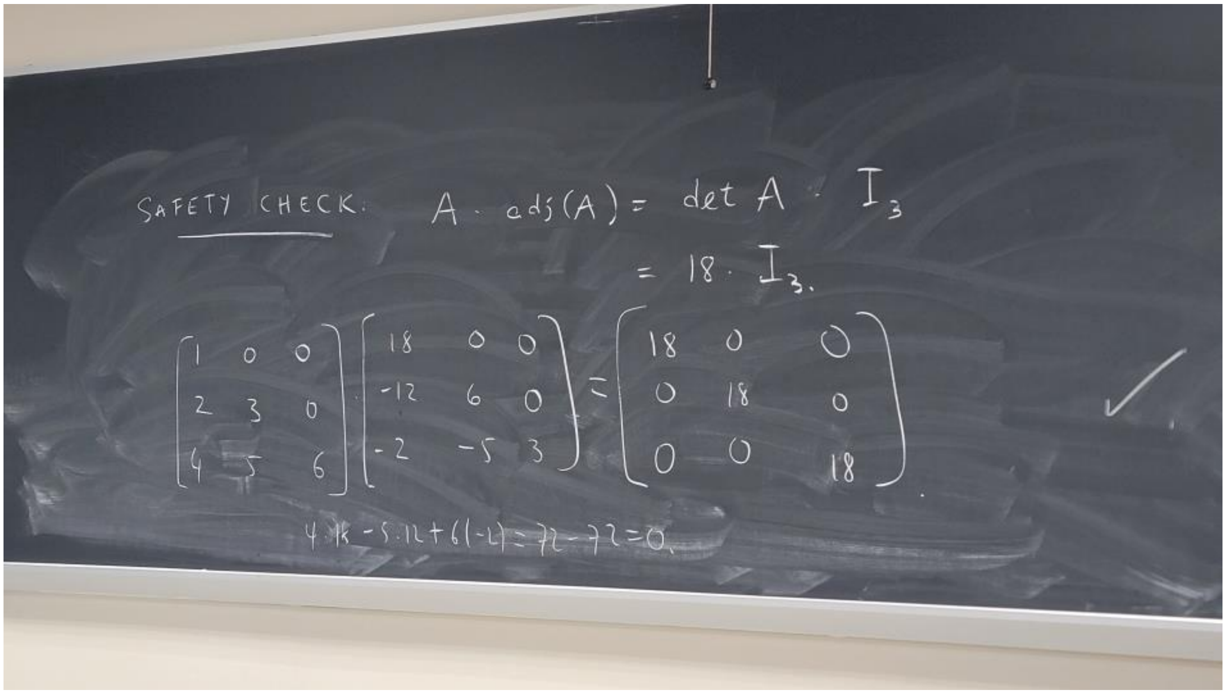

DEFINITION: GIVEN AN $n \times n$ MATRIX A,
A VECTOR $\underline{v} \in \mathbb{R}^n$ WHICH SOLVES THE
EQUATION $A\underline{v} = \lambda \underline{v}$, $\lambda \in \mathbb{R}$
IS CALLED AN EIGENVECTOR OF A WITH EIGENVALUE λ.

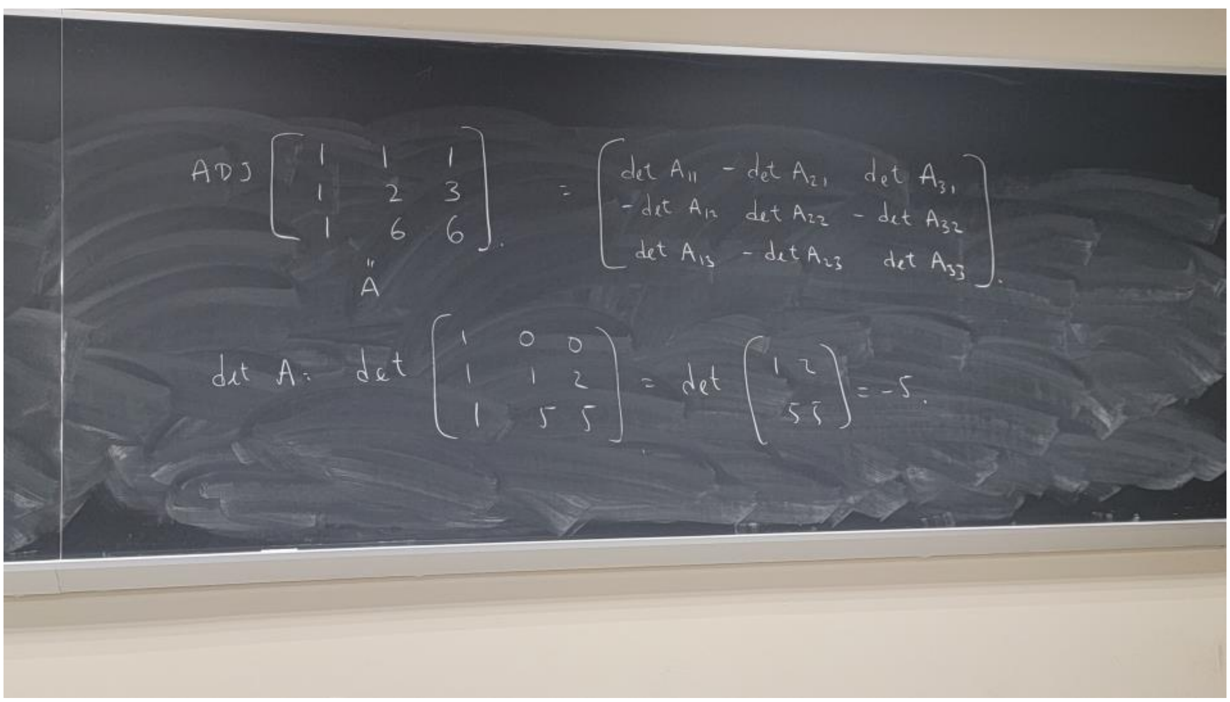

THEOREM: AN $n \times n$ MATRIX A IS DIAGONALIZABLE
IF AND ONLY IF THERE IS A BASIS $\underline{v}_1, \dots, \underline{v}_n$
FOR $\mathbb{R}^n$ OF EIGENVECTORS OF A
$A\underline{v}_i = \lambda_i \underline{v}_i$.

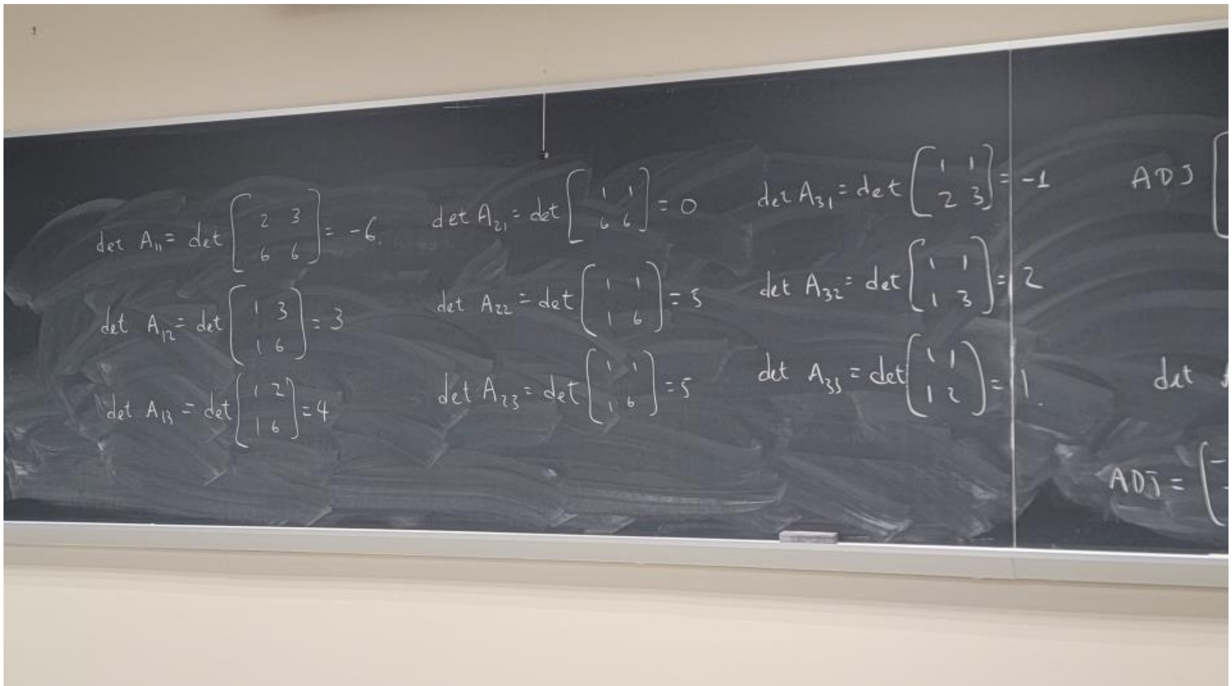

EXAMPLES.
ADJ $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$.
THIS IS THE MATRIX $B = (b_{ij})$, $b_{ij} = (-1)^{i+j} \det(A_{ji})$.
$= \begin{pmatrix} \det A_{11} & -\det A_{21} & \det A_{31} \\ -\det A_{12} & \det A_{22} & -\det A_{32} \\ \det A_{13} & -\det A_{23} & \det A_{33} \end{pmatrix}$.


det A11 = det [3 0; 5 6] = 18
det A12 = det [2 0; 4 6] = 12.
det A13 = det [2 3; 4 5] = -2
det A21 = det [0 0; 5 6] = 0
det A22 = det [1 0; 4 6] = 6.
det A23 = det [1 0; 4 5] = 5
det A31 = det [0 0; 3 0] = 0
det A32 = det [1 0; 2 0] = 0
det A33 = det [1 0; 2 3] = 3


A D J [1 0 0; 2 3 0; 4 5 6]
det = 18
ADJ A = [18 0 0; -12 6 0; -2 -5 3]


SAFETY CHECK: $A \cdot adj(A) = \det A \cdot I_3$
$= 18 \cdot I_3.$
$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 18 & 0 & 0 \\ -12 & 6 & 0 \\ -2 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}.$
$4 \cdot 18 - 5 \cdot 12 + 6(-2) = 72 - 72 = 0.$


$\text{ADJ} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 6 & 6 \end{bmatrix}}_{= A} = \begin{bmatrix} \det A_{11} & -\det A_{21} & \det A_{31} \\ -\det A_{12} & \det A_{22} & -\det A_{32} \\ \det A_{13} & -\det A_{23} & \det A_{33} \end{bmatrix}.$
$\det A = \det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 5 & 5 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 5 & 5 \end{bmatrix} = -5.$


$\det A_{11} = \det \begin{bmatrix} 2 & 3 \\ 6 & 6 \end{bmatrix} = -6.$
$\det A_{21} = \det \begin{bmatrix} 1 & 1 \\ 6 & 6 \end{bmatrix} = 0$
$\det A_{31} = \det \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = -1$
ADJ
$\det A_{12} = \det \begin{pmatrix} 1 & 3 \\ 1 & 6 \end{pmatrix} = 3$
$\det A_{22} = \det \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix} = 5$
$\det A_{32} = \det \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = 2$
$\det A_{13} = \det \begin{pmatrix} 1 & 2 \\ 1 & 6 \end{pmatrix} = 4$
$\det A_{23} = \det \begin{pmatrix} 1 & 1 \\ 1 & 6 \end{pmatrix} = 5$
$\det A_{33} = \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 1.$
det
ADJ =